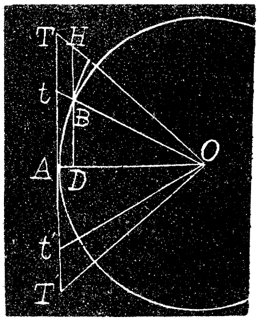
ISOPERIMETRY WITHOUT CURVES OR CALCULUS.

By PROFESSOR P. H. PHILBRICK, M. Sc., C. E., Lake Charles. La.

[Continued from the November Number.]

PROPOSITION IX. *If two regular polygons have the same perimeter, the one having the greater number of sides has the greatest area.*

Describe a circle with any radius $AO=r$ and circumference $2\pi r=c$.



Take AB and AC respectively the mth and nth part of a semi-circumference, and draw the secants OBt and OCT to meet the tangent AtT. Draw also DBH parallel to AT, and the tangent Be. Then, At is one-half of one side of a regular polygon of m sides, whose apothem is AO; and AT is one-half of one side of a regular polygon of n sides, whose apothem is likewise AO.

Let P=the perimeter of the polygon, the length of each side of which is $2AT$, and p=the perimeter of the polygon, the length of each side of which is $2At$.

Let a=the arc AB and A=the arc AC.

$$\text{Then, } P=AT\frac{c}{A},\ \ p=At\ \frac{c}{a},\ \therefore\ \ \frac{P}{p}=\frac{AT}{At}\cdot\frac{a}{A}.$$

Now, $BH>Be>$arc BC and $BD<$ arc AB.

$$\text{Dividing gives, } \frac{BH}{BD}>\frac{\text{arc } BC}{\text{arc } AB} \text{ or } \frac{DH}{DB}>\frac{\text{arc } AC}{\text{arc } AB}=\frac{A}{a}.$$

$$\text{But } \frac{AT}{At}=\frac{DH}{DB} \text{ and therefore } \frac{AT}{At}>\frac{A}{a}.$$

$$\text{Multiplying by} \frac{a}{A} \text{ we have, } \frac{AT}{At}\cdot\frac{a}{A}>1.$$

$$\text{Therefore, } \frac{P}{p}>1 \text{ or } P>p.$$

Hence, for the same apothem, the perimeter of the polygon of the greater number of sides is the smaller.

Therefore, for *equal* perimeters, the apothem of the polygon of the greater number of sides must be the greater; and since, for equal perimeters, the areas vary as the apothem, the area of the polygon having the greater number of sides is likewise the greater.

PROPOSITION X. *If two regular polygons have the same area, the one having the greater number of sides has the least perimeter.*

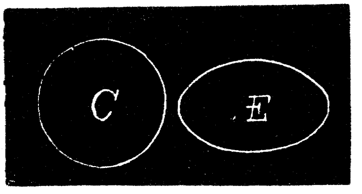
If the perimeters were equal, then (Prop. IX) the area of the one hav-

ing the greater number of sides would be the greater. Hence, since the areas are equal, the perimeters of the polygon having the greater number of sides is the smaller.

PROPOSITION XI. *Of all isoperemetric figures, the circle has the maximum area.*

We prove by (Prop. VIII) that of regular isoperemetric polygons, that having the greatest number of sides has the greatest area, and hence if the number of sides of any regular polygon be continually increased, keeping its perimeter the same, its area will be continually increased; and as the circle is the limiting figure in conformity to which the regular polygon continually approaches, as the number of its sides is made greater and greater, the circle is that figure, which for a given perimeter contains the maximum area.

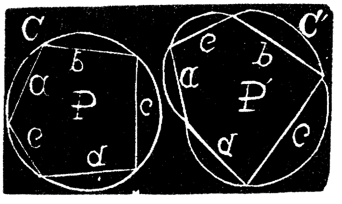
PROPOSITION XII. *Of all plane figures containing the same area, the circle has the minimum perimeter.*



Let C be a circle and E any other figure having the same area as C.

Now, by Prop. XI, if the perimeter of C was equal to that of E its area would be greater than that of E; but since it is the same, its perimeter must be less than that of E.

PROPOSITION XIII. *Of all polygons formed with the same given sides, that which can be inscribed in a circle is a maximum.*



Let the polygon P, having the sides a, b, c, d, and e, be inscribed in a circle, and the polygon P' formed with the same sides, not be inscriptible.

Upon the sides a, b, c, etc., of the polygon P' construct circular segments, equal to those standing upon the corresponding sides of the polygon P.

Then the whole figure C' thus found, has the same perimeter as the circle C.

Hence, (Prop. XI) area of $C >$ area of C'; and subtracting the circular segments from both, we have, $P > P'$.

[Concluded.]